STUDY GUIDE

Rational Functions

Precalculus 1.8 - 1.10

End Behavior, Asymptotes, Holes & Equivalent Representations

Overview

Lessons 1.8-1.10 build on each other into one skill: given a rational function

$y = f(x) / g(x)$ (f and g are both polynomials, $g(x) \neq 0$)

you should be able to find **every** key feature of its graph - end behavior, vertical asymptotes, holes, x-intercepts, and slant asymptotes - and go the other direction too, writing an equation for a function when you're given its features or its graph. 1.10 is really just 1.8 + 1.9 combined into one repeatable procedure, so master those two first.

Throughout this guide, look for boxes like this one - stop and answer them yourself *before* reading the answer that follows:

> ***Quick check:*** *What is 7 + 8? →* ***Answer:*** *15. (See - that's all these are: a one-step pause to make sure the idea that just got explained actually landed, before moving on to the next one.)*

1.8 - Rational Functions and End Behavior

Rational function: the quotient of two polynomials, $y = f(x)/g(x)$, where $g(x) \neq 0$. Because polynomial functions have domain "all real numbers," but rational functions divide by another polynomial, the domain of a rational function excludes any x-value that makes the denominator zero.

> ***Quick check:*** *Is $y = (x + 1)/5$ a rational function in the sense this lesson cares about (the kind with domain restrictions)? →* ***Answer:*** *No - the "denominator" here is just the constant 5, which is never zero, so this behaves like an ordinary polynomial with no restricted domain. The interesting cases are when the denominator is a polynomial with a variable in it.*

End behavior: compare the degrees of the leading terms

Write the function using just its leading terms, $f(x) = a \cdot x^n / (b \cdot x^d)$, where n is the degree of the numerator and d is the degree of the denominator. There are exactly three cases:

Case	Degree comparison	End behavior
I	n = d (same degree)	Horizontal asymptote: **y = a/b** (ratio of leading coefficients)
II	n < d (denominator's degree wins)	Horizontal asymptote: **y = 0**
III	n > d (numerator's degree wins)	No horizontal asymptote. f(x) behaves like the polynomial $y = (a/b)x^{n-d}$ as $x \to \pm\infty$

> ***Quick check:*** *If n = 4 and d = 4, which case are you in, and if the leading coefficients are 9 and 3, what's the horizontal asymptote? →* ***Answer:*** *Case I (same degree) → y = 9/3 =* ***y = 3****.*
>
> ***Quick check:*** *If n = 2 and d = 5, which case, and what's the horizontal asymptote - no need to even know the actual coefficients this time. →* ***Answer:*** *Case II (denominator degree wins) →* ***y = 0****, regardless of what the leading coefficients are.*

Special case of III - the slant (oblique) asymptote: if the numerator's degree is *exactly one more* than the denominator's degree (n = d + 1), the function's end behavior is a straight line, not a curve. That line is called a slant asymptote.

> ***Quick check:*** *n = 5, d = 4. Does this function have a horizontal asymptote, a slant asymptote, or neither? →* ***Answer:*** *n is exactly one more than d (5 = 4 + 1), so* ***slant asymptote*** *- not "no asymptote at all." That phrase only applies once the gap is 2 or more.*

Describing end behavior with limit notation

Once you know the horizontal asymptote (or the polynomial the function behaves like), write it as two limit statements:

$$\lim_{x \to -\infty} f(x) = [\text{value}] \qquad \lim_{x \to \infty} f(x) = [\text{value}]$$

If there's a horizontal asymptote y = k, both limits equal k. If the function grows or shrinks without bound instead, the limit is ∞ or -∞.

Worked Example 1

Determine whether each function has a horizontal asymptote, a slant asymptote, or neither. If it has a horizontal asymptote, state it.

a) $f(x) = (4x^2 - x + 6) / (2x^2 + 5)$

Both leading terms are degree 2 (Case I): H.A. → $y = 4/2 =$ **$y = 2$**

b) $g(x) = (3x + 1) / (x^2 - 4)$

Numerator degree 1 < denominator degree 2 (Case II): **$y = 0$**

c) $h(x) = (5x^3 + 2x - 1) / (x^2 + 9)$

Numerator degree 3 > denominator degree 2, and 3 is exactly one more than 2, so this is a **slant asymptote** case, not Case III's general "no asymptote." (See Example 2 for finding its equation.)

Worked Example 2 - finding a slant asymptote

Find the slant asymptote of $f(x) = (x^2 + x - 8) / (x - 2)$.

Use polynomial (or synthetic) long division and ignore the remainder:

$x^2 + x - 8 \div (x - 2) \rightarrow$ quotient: $x + 3$, remainder: -2

Slant asymptote: $y = x + 3$

> ***Quick check:*** *After dividing, you get a quotient of $2x - 1$ with a remainder of 7. What's the slant asymptote? →* ***Answer: $y = 2x - 1$****. The remainder never appears in the asymptote - only the quotient does.*

Worked Example 3 - limit statements from a table

A function g(x) has these values:

x	-1000	-100	100	1000
g(x)	4.98	4.8	5.2	5.02

As $x \to \pm\infty$, g(x) is settling toward 5 from both directions, so:

$\lim_{x\to-\infty} g(x) = 5 \qquad \lim_{x\to\infty} g(x) = 5$

1.9 - Vertical Asymptotes and Removable Discontinuities

Vertical asymptotes and holes **both** happen at x-values that make the denominator zero - the difference is entirely about whether that factor also appears in the numerator.

	A hole occurs when...	A vertical asymptote occurs when...
Rule	the factor in the denominator **cancels** with a matching factor in the numerator	the factor in the denominator **cannot cancel** with anything in the numerator
What the graph does	a single open circle at that x-value; the function is still approaching a real y-value from both sides	the graph shoots off toward $+\infty$ or $-\infty$ on at least one side
Limit notation	lim f(x) = [a real number]	lim f(x) = $\pm\infty$

> ***Quick check:*** *f(x) = [(x - 4)(x + 2)] / [(x + 2)(x - 6)]. The factor (x + 2) appears in both the top and bottom. Hole or vertical asymptote - and at what x-value? →* ***Answer:*** *It cancels, so it's a* ***hole, at x = -2****. (The leftover (x - 6) in the denominator is what gives the vertical asymptote instead, at x = 6.)*

Finding the coordinates of a hole: cancel the common factor first, then plug the x-value into what's *left over* (the simplified function) - not the original. Plugging into the original just gives 0/0.

> ***Quick check:*** *After canceling, f(x) simplifies to (x + 3)/(x - 1), and the hole is at x = 2. What's the hole's y-coordinate? →* ***Answer:*** *Plug x = 2 into the simplified form: (2+3)/(2-1) = 5/1 =* ***5****, so the hole is at* ***(2, 5)****.*

One-sided limits at a vertical asymptote: state the left-hand and right-hand limit separately, since the graph can go to $+\infty$ on one side and $-\infty$ on the other:

$\lim_{x\to c^-} f(x) = ?$ $\lim_{x\to c^+} f(x) = ?$

Worked Example 4

For f(x) = [(x - 5)(x + 1)] / [(x + 1)(x - 3)], find any holes and vertical asymptotes.

The factor (x + 1) appears in both numerator and denominator → cancels → **hole at x = -1**. The factor (x - 3) in the denominator has nothing to cancel with → set it to zero: x - 3 = 0 → **vertical asymptote at x = 3**.

Coordinate of the hole: simplify first → f(x) = (x - 5)/(x - 3), then plug in x = -1:

f(-1) = (-1 - 5)/(-1 - 3) = -6/-4 = 3/2

Hole at (-1, 3/2)

Worked Example 5 - writing an equation from given features

Write an equation for a rational function f(x) where:

$\lim_{x\to 2^-} f(x) = 6$, $\lim_{x\to 2^+} f(x) = 6$ (hole at (2, 6)) $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to 4^+} f(x) = \infty$ (V.A. at x = 4)

Start with the two known zeros of the denominator as factors, then solve for the leading coefficient a using the hole's y-value:

f(x) = a(x - 2) / [(x - 2)(x - 4)]

Plug the hole's x-value into the *reduced* form a/(x - 4) and set it equal to 6:

a/(2 - 4) = 6 → a = -12

f(x) = -12(x - 2) / [(x - 2)(x - 4)]

> ***Quick check:*** *Same setup, but the hole is at (2, 6) and the vertical asymptote is at $x = 4$ - if instead the hole's y-value were 3 (not 6), what would a equal?* → ***Answer:*** *$a/(2 - 4) = 3$ → $a/(-2) = 3$ →* ***a = -6****.*

Worked Example 6 - full analysis

Find all key features of $g(x) = (2x^2 - 2) / (x^2 + x - 2)$ and describe the graph.

Factor: $g(x) = 2(x-1)(x+1) / [(x+2)(x-1)]$

- Common factor $(x - 1)$ cancels → **hole at $x = 1$**. Simplified: $g(x) = 2(x+1)/(x+2)$. Hole y-value: $2(1+1)/(1+2) = 4/3$ → **hole at $(1, 4/3)$**
- Remaining denominator factor $(x + 2)$ doesn't cancel → **vertical asymptote at $x = -2$**
- Same degree top and bottom (both degree 2 originally) → **horizontal asymptote $y = 2/1 = 2$**
- x-intercept: numerator zero that isn't the hole → $x = -1$ → **$(-1, 0)$**

1.10 - Equivalent Representations

This lesson is about using **factored form** as the master tool that answers every question at once: zeros, domain, holes, and asymptotes. Here's the procedure to run on any rational function.

The full-analysis procedure

1. **Factor** the numerator and denominator completely.
2. **Cancel** any common factors - each one you cancel is a **hole**. Keep track of what you canceled; you'll need it to find the hole's y-coordinate later.
3. Whatever's left in the denominator, set each factor to zero → these are your **vertical asymptotes**.
4. Whatever's left in the numerator, set each factor to zero → these are your **x-intercepts** (zeros).
5. Set x = 0 in the *original* (or simplified - they agree unless x = 0 is the hole) to get the **y-intercept**.
6. Compare degrees of the *original, unfactored* numerator and denominator for **end behavior** (horizontal asymptote, slant asymptote, or neither - see 1.8).
7. State the **domain** as all reals except every x-value found in steps 2 and 3.

> ***Quick check:*** *You're on step 3 and, after canceling, you find the factor (x + 9) still sitting in the denominator. What do you write down, and for what x-value? →* ***Answer:*** *A* ***vertical asymptote at x = -9*** *(set x + 9 = 0 and solve - the asymptote is at the zero of the factor, not at "-9's opposite" or the factor itself).*

Worked Example 7 - the full procedure together

Let $k(x) = (x^2 + 2x - 3) / (x^2 - x - 6)$.

1. Factor: $k(x) = (x+3)(x-1) / [(x-3)(x+2)]$
2. Nothing cancels → **no holes**
3. Vertical asymptotes: x - 3 = 0 and x + 2 = 0 → **x = 3, x = -2**
4. x-intercepts: x + 3 = 0 and x - 1 = 0 → **(-3, 0)** and **(1, 0)**
5. y-intercept: k(0) = (0+0-3)/(0-0-6) = -3/-6 = 1/2 → **(0, 1/2)**
6. Same degree (2 and 2) → **H.A. y = 1**
7. Domain: **(-∞, -2) ∪ (-2, 3) ∪ (3, ∞)**

Worked Example 8 - writing an equation from a graph

A graph shows a hole at x = 3, a vertical asymptote at x = -1, and passes through the point (0, 2). Write a possible equation.

Set up the skeleton from the hole and the asymptote (both become factors that appear in the denominator; the hole's factor also appears in the numerator):

$f(x) = a(x - 3) / [(x - 3)(x + 1)]$

Use the given point to solve for a. Plug into the *reduced* form a/(x+1) at x = 0:

$a/(0 + 1) = 2 \;\; \rightarrow \;\; a = 2$

$f(x) = 2(x - 3) / [(x - 3)(x + 1)]$

Recognizing horizontal vs. slant vs. neither at a glance

Compare the **original, unfactored** degree of numerator (n) and denominator (d) - factoring can hide a cancellation that changes what the graph *looks like* near a hole, but end behavior only ever depends on the original degrees:

Compare n and d	Result
n = d	Horizontal asymptote, y = ratio of leading coefficients
n < d	Horizontal asymptote, y = 0

Compare n and d	Result
n = d + 1	Slant asymptote (do the long division)
n > d + 1	Neither H.A. nor S.A. - end behavior curves like a higher-degree polynomial

> ***Quick check:*** *A function's original numerator has degree 6, and its original denominator has degree 4 - even though 3 different factors cancel when you simplify it. Does it have a horizontal asymptote, a slant asymptote, or neither? →* ***Answer: Neither.*** *The gap between 6 and 4 is 2, which is more than 1 - and this is decided by the original degrees, before any canceling, exactly because canceling doesn't change how fast the top and bottom actually grow.*

Quick Reference Card

Feature	How to find it
Domain	All reals except zeros of the (fully simplified, before canceling) denominator
Hole	A factor that cancels between numerator and denominator. Plug the x-value into the *simplified* function for the y-coordinate.
Vertical asymptote	A factor left in the denominator after canceling. Set it to zero.
x-intercept	A factor left in the numerator after canceling. Set it to zero.
y-intercept	Evaluate the function at $x = 0$
Horizontal asymptote	Compare degrees of the *original* numerator/denominator - same degree → ratio of leading coefficients; numerator smaller → $y = 0$
Slant asymptote	Numerator's degree is exactly 1 more than denominator's - do polynomial long division, ignore the remainder
Neither H.A. nor S.A.	Numerator's degree is 2 or more higher than the denominator's

Common Mistakes to Avoid

- **Plugging into the original function to find a hole's y-coordinate.** This always gives 0/0. Always simplify (cancel) first, *then* plug in.
- **Forgetting to check the original degree for end behavior after factoring.** A canceled factor doesn't change the total degree of the numerator or denominator - use the degrees you started with.
- **Mixing up which direction "n" and "d" go.** n is always the numerator's degree, d is always the denominator's. $n < d$ means the *bottom* grows faster, pulling the function down toward $y = 0$.
- **Forgetting the slant asymptote's remainder gets dropped**, not added back in. Only the quotient part of the long division is the asymptote.
- **Writing "no asymptote" for every degree mismatch.** Only $n > d$ **by more than 1** has no H.A./S.A. at all; $n = d + 1$ specifically gives a slant asymptote.
- **Stating a vertical asymptote at the factor itself instead of where it equals zero.** A denominator factor of $(x + 5)$ gives a vertical asymptote at $x =$ **-5**, not $x = 5$.

Practice Problems

1. Determine the horizontal asymptote (or state "slant" or "neither"): $f(x) = (7x^2 + 3x) / (2x^2 - 9)$

2. Determine the horizontal asymptote (or state "slant" or "neither"): $g(x) = (x + 4) / (x^3 - 2x + 1)$

3. Find the slant asymptote: $h(x) = (x^2 - 5x + 1) / (x - 1)$

4. Write limit statements (as $x \to -\infty$ and $x \to \infty$) for a function with horizontal asymptote $y = -2/5$.

5. For $f(x) = [(x + 6)(x - 2)] / [(x - 2)(x + 4)]$, identify the hole and the vertical asymptote.

6. Find the coordinates of the hole in problem 5.

7. Write the left- and right-hand limit statements as x approaches the vertical asymptote you found in problem 5, given that the graph goes to $+\infty$ on the left of the asymptote and $-\infty$ on the right.

8. A rational function has a hole at (5, -1/2) and a vertical asymptote at $x = 2$. Write a possible equation for it.

9. For $k(x) = (x^2 - 9) / (x^2 + x - 12)$, find: the factored form, any holes, any vertical asymptotes, the x-intercept(s), the y-intercept, and the horizontal asymptote.

10. A graph has a hole at $x = -3$, a vertical asymptote at $x = 1$, and passes through the point (0, -6). Write a possible equation in factored form.

11. Classify each as having a horizontal asymptote, a slant asymptote, or neither: a) $y = (5x^3 - x) / (x^2 + 4)$ b) $y = (3x^2 + x - 1) / (6x^2 - 5)$ c) $y = (x^4 + 2x) / (x - 3)$

12. For $d(x) = [2x(x - 1)(x + 3)] / [(x - 1)^2(x + 5)]$, how many holes does the graph have, and how many vertical asymptotes? (Careful - a squared factor behaves differently than a single factor once one copy of it cancels.)

Answer Key

1. Same degree (2 = 2) → **H.A. y = 7/2**

2. Numerator degree 1 < denominator degree 3 → **H.A. y = 0**

3. $x^2 - 5x + 1 \div (x - 1)$ → quotient x - 4 (remainder -3, ignored) → **y = x - 4**

4. $\lim_{x\to-\infty} f(x) = -2/5$ $\lim_{x\to\infty} f(x) = -2/5$

5. (x - 2) cancels → **hole at x = 2**. Remaining denominator factor (x + 4) → **V.A. at x = -4**

6. Simplify to f(x) = (x+6)/(x+4), plug in x = 2: (2+6)/(2+4) = 8/6 = 4/3 → **hole at (2, 4/3)**

7. $\lim_{x\to-4^-} f(x) = \infty$ $\lim_{x\to-4^+} f(x) = -\infty$

8. Skeleton: a(x - 5)/[(x-5)(x-2)]. Plug hole's x = 5 into reduced form a/(x-2) and set equal to -1/2: a/(5-2) = -1/2 → a = -3/2. **f(x) = -3/2(x - 5) / [(x - 5)(x - 2)]** (any nonzero multiple of this also works)

9. Factored: k(x) = (x-3)(x+3) / [(x+4)(x-3)]. Common factor (x-3) cancels → **hole at x = 3**. Simplified: (x+3)/(x+4). Hole y-value: (3+3)/(3+4) = 6/7 → **hole at (3, 6/7)**. Remaining denominator factor → **V.A. at x = -4**. Remaining numerator factor → **x-intercept (-3, 0)**. y-intercept: k(0) = (0-9)/(0-12) = 3/4 → **(0, 3/4)**. Same degree (2 = 2) → **H.A. y = 1**.

10. Skeleton: a(x+3)/[(x+3)(x-1)]. Plug x = 0 into reduced form a/(x-1): a/(0-1) = -6 → a = 6. **f(x) = 6(x + 3) / [(x + 3)(x - 1)]**

11. a) numerator degree 3, denominator degree 2, difference of 1 → **slant asymptote** b) same degree (2 = 2) → **horizontal asymptote, y = 3/6 = 1/2** c) numerator degree 4, denominator degree 1, difference of 3 → **neither**

12. One copy of (x - 1) cancels between numerator and denominator, but the denominator had $(x - 1)^2$ - *two* copies - so one copy is still left behind in the denominator after canceling, along with (x + 5). That means x = 1 is **not** a hole after all - it's a **vertical asymptote**, because a factor only becomes a hole if it disappears *completely*. **Holes: 0. Vertical asymptotes: 2 (x = 1 and x = -5).** This is exactly the kind of case the Quick Checks above are trying to build a reflex for - always check whether the *whole* factor is gone, not just whether you got to cross something out once.